

SUPPORTS OF WEIGHTED EQUILIBRIUM MEASURES AND EXAMPLES

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ABSTRACT. We analyze the supports of weighted equilibrium measures in \mathbb{C}^n . We give explicit examples of families of compact sets which arise as the support of a weighted equilibrium measure for some admissible weight w . These examples also give new constructions of plurisubharmonic functions in the Lelong class. We also include a list of open problems on the support of extremal measures which are related to solutions of Monge-Ampère equations.

1. INTRODUCTION AND BACKGROUND

Determining the support S_w of the Monge-Ampère measure of a weighted extremal function $V_{K,Q}$ is important for many reasons. Firstly, the domination principle (Theorem 2.2) shows the importance of the support of the Monge-Ampère measure of a weighted extremal function. Namely, we have $V_{K,Q} = V_{S_w,Q|_{S_w}}$. Secondly, for a continuous weight w on a closed set K , the weighted supremum norm of a polynomial p_d of degree less than or equal to d is attained on the support of the weighted extremal measure i.e., $\|w^d p_d\|_K = \|w^d p_d\|_{S_w}$.

Next, a weighted approximation is possible only on the support of the Monge-Ampère measure of the weighted extremal function. Namely, if $f \in \mathcal{C}(K)$, then there exists a weighted sequence $(w^d p_d)_{d=1}^\infty$ converging uniformly to f on K only if $f(z) = 0$ for all $z \in K \setminus S_w$. Some references for the weighted approximation are [5], [13] and [15].

In one variable, there is a notion of logarithmic capacity associated to a compact set K . In order to estimate the weighted analogue of the capacity with respect to an admissible weight w on a compact set K numerically, it is useful to determine the support of the weighted extremal measure. Using the fact that the weighted capacity of a closed set is equal to the weighted capacity of the support of the weighted extremal measure, one may reduce the calculation time remarkably, e.g., compare Table 4 and Table 5 in [12]. See [12], [13] for further discussion of weighted capacities.

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This paper has two motivations. The first motivation is to further analyze the supports of weighted equilibrium measures and to extend the results from $n = 1$ and from the unweighted setting to the weighted setting in higher dimensions. The second one is to provide families of examples of compact sets which arise as supports of weighted equilibrium measures. Furthermore, some of these examples give new ways of constructing plurisubharmonic functions in the Lelong class.

In the second section, we recall some of the basic definitions and necessary facts from weighted and classical pluripotential theories. We also demonstrate by a counterexample that in the weighted setting, unlike the unweighted setting, the weighted extremal measure and the weighted relative extremal measures are not necessarily mutually absolutely continuous. In the third section, we give several families of examples of compact sets that are supports of weighted extremal measures. In particular, to construct the radial extremal functions we survey Persson's representation of radial plurisubharmonic functions in terms of their Monge-Ampère measures. In Theorem 3.3, which is the main theorem and the deepest result, we construct extremal functions whose Monge-Ampère measures are supported on the closure of strictly pseudoconvex domains. In the last section, we discuss some open problems related to the subject.

2. WEIGHTED PLURIPOTENTIAL THEORY

We recall basic definitions and facts from weighted and classical pluripotential theories. We refer to Saff and Totik's book [13] for $n = 1$ and Thomas Bloom's Appendix B of [13] for $n > 1$.

Let K be a closed subset of \mathbb{C}^n . A function $w : K \rightarrow [0, \infty)$ is called an **admissible weight function** on K if

- i) w is upper semicontinuous.
- ii) The set $\{z \in K \mid w(z) > 0\}$ is not pluripolar.
- iii) If K is unbounded, $|z|w(z) \rightarrow 0$ as $|z| \rightarrow \infty$, $z \in K$.

We define $Q = -\log w$. We use Q and w interchangeably.

The **weighted extremal function** of K with respect to Q is defined as

$$(2.1) \quad V_{K,Q}(z) := \sup \{u(z) \mid u \in L, u \leq Q \text{ on } K\},$$

where the Lelong class L is defined as

$$(2.2) \quad L := \{u \mid u \in PSH(\mathbb{C}^n), u(z) \leq \log^+ |z| + C\},$$

where C depends on u . Here the set of all plurisubharmonic functions on a domain Ω is denoted by $PSH(\Omega)$. If $Q = 0$, then we call it the

(**unweighted**) **extremal function** of K and denote it by V_K . A compact set K is called **regular** if V_K is continuous. If $K \cap \overline{B(z, r)}$ is regular for all $z \in K$ and $r > 0$, the set K is called **locally regular**. Here, $B(z, r)$ denotes the open ball of radius r and center z .

We note that the **upper semicontinuous regularization** of a function v is defined by $v^*(z) := \limsup_{w \rightarrow z} v(w)$ and it is well known that the upper semicontinuous regularization of $V_{K,Q}$ is plurisubharmonic and in L^+ where

$$L^+ := \{u \in L \mid \log^+ |z| + C \leq u(z)\},$$

where C depends on u .

We recall that $dd^c u = 2i\partial\bar{\partial}u$ and $(dd^c u)^n$ is the **complex Monge-Ampère operator** defined by $(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u$ for plurisubharmonic functions which are \mathcal{C}^2 . If u is a locally bounded plurisubharmonic function, then $(dd^c u)^n$ is defined as a positive measure. See [8] for the details. It is also well known that $\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n$ for all $u \in L^+$.

A set E is called **pluripolar** if $E \subset \{z \in \mathbb{C}^n \mid u(z) = -\infty\}$ for some plurisubharmonic function u . If a property holds everywhere except on a pluripolar set we will say that the property holds **quasi everywhere**. It is a well-known fact that $V_{K,Q} = V_{K,Q}^*$ quasi everywhere.

We denote the support of $(dd^c V_{K,Q}^*)^n$ by S_w . The following lemma is very useful to determine the supports of Monge-Ampère measures.

Lemma 2.1. [13, Appendix B, Lemma 2.3] *Let $S_w^* := \{z \in \mathbb{C}^n \mid V_{K,Q}^*(z) \geq Q(z)\}$. Then, we have $S_w \subset S_w^*$.*

The following theorem is a very important tool in pluripotential theory and we will use it frequently to determine weighted extremal functions.

Theorem 2.2. (Domination Principle.) [2, Lemma 6.5] *If $u \in L$, $v \in L^+$ and if $u \leq v$ holds almost everywhere with respect to $(dd^c v)^n$, then $u \leq v$ on \mathbb{C}^n .*

Definition 2.3. A plurisubharmonic function u on an open set Ω is called **maximal** if for any relatively compact open subset ω of Ω and any upper semicontinuous function v defined on $\bar{\omega}$ which is plurisubharmonic on ω such that $v \leq u$ on the boundary of ω , then $v \leq u$ in ω .

The following theorem characterizes maximal plurisubharmonic functions in terms of their Monge-Ampère measures.

Theorem 2.4. [8, Theorem 4.4.2] *Let Ω be an open subset of \mathbb{C}^n and u be a locally bounded plurisubharmonic function on Ω . Then, $(dd^c u)^n = 0$ if and only if u is maximal.*

The following theorem is called **Poisson Modification** and it is used to modify a locally bounded plurisubharmonic function in a ball to obtain another locally bounded plurisubharmonic function such that the new function is maximal in the ball and equal to the original function outside the ball.

Theorem 2.5. [13, Appendix B, Theorem 1.3] *Let $u \in PSH(\Omega) \cap L_{loc}^\infty(\Omega)$. Let $z_0 \in \Omega$ and $R > 0$ such that $\overline{B(z_0, R)} \subset \Omega$. Then there exists $\tilde{u} \in PSH(\Omega) \cap L_{loc}^\infty(\Omega)$ such that*

- i) $(dd^c \tilde{u})^n = 0$ on $B(z_0, R)$;
- ii) $\tilde{u} = u$ on $\Omega \setminus B(z_0, R)$;
- iii) $\tilde{u} \geq u$ on $B(z_0, R)$.

We recall the definition of weighted extremal functions relative to an open subset of \mathbb{C}^n .

Definition 2.6. Let Ω be a bounded domain in \mathbb{C}^n and K be a compact subset of Ω . Let $Q : K \rightarrow (-\infty, 0]$ be a function on K . The **weighted relative extremal function** of K with respect to Ω and weight Q is defined as

$$(2.3) \quad U_{K,Q,\Omega}(z) := \sup\{u(z) \mid u \in PSH(\Omega), u < 0 \text{ on } \Omega, u \leq Q \text{ on } K\}.$$

It is well known that the **regularized weighted relative extremal function** of K with respect to Ω and weight Q defined by

$$(2.4) \quad U_{K,Q,\Omega}^*(z) := \limsup_{w \rightarrow z} U_{K,Q,\Omega}(w)$$

is plurisubharmonic and maximal out of K . Again, we have $U_{K,Q,\Omega} = U_{K,Q,\Omega}^*$ quasi everywhere.

If $Q = -1$, then $U_{K,Q,\Omega}$ is called **(unweighted) relative extremal function** of K with respect to Ω , and we denote it by $U_{K,\Omega}$.

Here we give an analogue of Lemma 2.1 for relative extremal functions.

Proposition 2.7. *Let Q be an admissible weight on K (i.e., $\{z \in K \mid Q(z) < 0\}$ is not pluripolar), where K is a compact subset of a hyperconvex domain Ω . Then, $(dd^c U_{K,Q,\Omega}^*)^n$ is supported on the set $\{z \in K \mid U_{K,Q,\Omega}^*(z) \geq Q(z)\}$.*

Proof. Let z_0 be a point in K , such that $U_{K,Q,\Omega}^*(z_0) < Q(z_0) - \varepsilon$, for some positive ε . We will show that $U_{K,Q,\Omega}^*$ is maximal in a neighborhood U of z_0 .

By the facts that Q is lower semicontinuous and $U_{K,Q,\Omega}^*$ is upper semicontinuous, we have $\{z \in K \mid Q(z) > Q(z_0) - \varepsilon/2\}$ is open in K relative to

Ω and $\{z \in \Omega \mid U_{K,Q,\Omega}^*(z) < U_{K,Q,\Omega}^*(z_0) + \varepsilon/2\}$ open. So we may find a ball of radius r around z_0 such that

$$\sup_{z \in B(z_0, r)} U_{K,Q,\Omega}^*(z) < \inf_{z \in B(z_0, r) \cap K} Q(z).$$

Applying the Poisson modification (Theorem 2.5) to $U_{K,Q,\Omega}^*$ on $B(z_0, r)$, we can find a plurisubharmonic function u such that $u \geq U_{K,Q,\Omega}^*$ on $B(z_0, r)$ and $u = U_{K,Q,\Omega}^*$ on $\Omega \setminus B(z_0, r)$, which is negative on Ω , and maximal on $B(z_0, r)$. Now u is a competitor for the relative extremal function because $u(z) \leq \sup_{z \in B(z_0, r)} U_{K,Q,\Omega}^*(z) < \inf_{z \in B(z_0, r) \cap K} Q(z)$ for all $z \in B(z_0, r)$. Hence, $u \equiv U_{K,Q,\Omega}^*$. Therefore, we get $U_{K,Q,\Omega}^*$ is maximal in a neighborhood of z_0 . Hence, z_0 is not in the support of $(dd^c U_{K,Q,\Omega}^*)^n$. \square

Levenberg showed in [10] that $(dd^c V_K^*)^n$ and $(dd^c U_{K,\Omega}^*)^n$ are mutually absolutely continuous for a non-pluripolar compact set K . However, unlike the unweighted case, $(dd^c V_{K,Q}^*)^n$ and $(dd^c U_{K,Q,\Omega}^*)^n$ are not necessarily absolutely continuous in general.

Example 2.8. Let K be the closed unit ball, $\overline{B(0, 1)}$. We define a continuous weight on K by letting $Q(z) = \max[\log |z|, -\frac{1}{2}] - 1$. Let $\Omega_1 := B(0, 2)$ and $\Omega_2 := B(0, e)$. We have the weighted relative extremal functions

$$\begin{aligned} U_{K,Q,\Omega_1}(z) &= \max \left[\max \left[\log |z|, -\frac{1}{2} \right] - 1, \frac{\log |z|}{\log 2} - 1 \right], \\ U_{K,Q,\Omega_2}(z) &= \max \left[\log |z|, -\frac{1}{2} \right] - 1, \end{aligned}$$

and the weighted (global) extremal function

$$V_{K,Q}(z) = \max \left[\log |z|, \frac{1}{2} \right] - 1.$$

Now the weighted relative extremal measure with respect to Ω_1 is supported on two concentric spheres. Namely, $\text{supp}(dd^c U_{K,Q,\Omega_1})^n = \partial B(0, 1) \cup \partial B(0, e^{-1/2})$. Whereas, $\text{supp}(dd^c V_{K,Q})^n = \text{supp}(dd^c U_{K,Q,\Omega_2})^n = \partial B(0, e^{-1/2})$. Therefore, $(dd^c U_{K,Q,\Omega_1})^n$ is not absolutely continuous with respect to $(dd^c V_{K,Q})^n$ and $(dd^c U_{K,Q,\Omega_2})^n$.

Example 2.9. Let K be the compact set $\overline{B(0, 1)}$. We define a continuous weight on K by letting $Q := \frac{1}{2} \max[\log |z|, -\frac{1}{2}] - \frac{1}{2}$. Let Ω be $B(0, e)$. Under these conditions, we have the weighted relative extremal function

$$U_{K,Q,\Omega}(z) = \frac{1}{2} \max \left[\log |z|, -\frac{1}{2} \right] - \frac{1}{2},$$

and the weighted (global) extremal function

$$V_{K,Q}(z) = \max \left[\frac{1}{2} \max \left[\log |z|, -\frac{1}{2} \right], \log |z| \right] - \frac{1}{2}.$$

In this case, the global (weighted) extremal measure is supported on two concentric spheres. Namely, $\text{supp}(dd^c V_{K,Q})^n = \partial B(0, 1) \cup \partial B(0, e^{-1/2})$. On the other hand, $\text{supp}(dd^c U_{K,Q,\Omega})^n = \partial B(0, e^{-1/2})$. Therefore, $(dd^c V_{K,Q})^n$ is not absolutely continuous with respect to $(dd^c U_{K,Q,\Omega})^n$.

3. EXAMPLES

In classical pluripotential theory, the extremal measures, $(dd^c V_K^*)^n$ and $(dd^c U_{K,\Omega}^*)^n$, of a compact set K are supported on the outer boundary of K . In the weighted setting, this does not hold true anymore. Many compact sets may arise as support of the Monge-Ampère measure of some weighted extremal function. In particular, by Theorem 4.1.1 of [13], for any compact set K in the plane, which has positive logarithmic capacity at every point of K (i.e., $C(K \cap B(z_0, r), \Omega) > 0$ for every $z_0 \in K$, for all $r > 0$ and for some hyperconvex domain Ω containing K), there exists an admissible weight on K such that the support of the weighted extremal measure is K . Unfortunately, the proof of the theorem utilizes the notion of logarithmic potentials which is not available in higher dimensions. Thus, determining the support of a weighted equilibrium measure and obtaining a similar result for higher dimensions are extremely difficult. In one dimension, Varju and Totik [15] found some necessary and sufficient conditions for weights on the unit circle that the support of the weighted extremal measure is the whole unit circle. In one variable, Benko, Damelin and Dragnev [3] also obtained some sufficiency conditions for the same problem. They also give explicit examples where the supports of the weighted equilibrium measures are the full circle or a finite union of arcs. Similar results for higher dimensions do not exist to our best knowledge. In this section, we show concrete examples of families of compact sets which are supports of some weighted extremal measures. Some references for supports of weighted equilibrium measures are [3], [5], [13] and [15].

The next proposition gives criteria for the weight Q such that the weighted extremal measure is supported on the boundary of K .

Proposition 3.1. *Let K be a closed set and Q is an admissible weight on K . Then $S_w \subset \partial K$, if either of the following holds:*

- (1) Q is maximal plurisubharmonic in the interior of K ,
- (2) Q is superharmonic in the interior of K .

Proof. We give the proof of the first. The proof of second part is similar. In the superharmonic case we use the fact that a plurisubharmonic function which is also superharmonic is in fact pluriharmonic.

Let Q be superharmonic in the interior of K . Note that Q is continuous in the interior of K since it is both upper semicontinuous and lower semicontinuous. Let $z_0 \in \text{int}(K)$, i.e., there exists $r > 0$ such that $B(z_0, r) \subset \text{int}(K)$. Since Q is continuous $V_{K,Q}^*(z) \leq Q(z)$ for all $z \in B(z_0, r)$. By applying the Poisson modification to the plurisubharmonic function $V_{K,Q}^*$ on the ball $B(z_0, r)$, we obtain $v \in L^+$ such that $v = V_{K,Q}^*$ on $\mathbb{C}^n \setminus B(z_0, r)$ and v is maximal on $B(z_0, r)$. Since $V_{K,Q}^* \leq Q$ on $B(z_0, r)$, we have $v \leq Q$ on $B(z_0, r)$. Hence v is a competitor for the weighted extremal function. Thus $v \equiv V_{K,Q}^*$, giving that $V_{K,Q}^*$ is maximal on $B(z_0, r)$. \square

Remark 3.2. Note that similar proofs give the same results of Proposition 3.1 for the weighted relative extremal measure.

This was an extreme case where the weighted extremal measure is supported on the boundary. Now another extreme case is that the closure of a strictly pseudoconvex domain can be obtained as the support of a weighted extremal measure.

Theorem 3.3. *Let Ω be a strictly pseudoconvex domain in \mathbb{C}^n with \mathcal{C}^2 boundary. There exists an admissible weight function Q on $K := \overline{\Omega}$ such that $\text{supp}(dd^c V_{K,Q}^*)^n = \overline{\Omega}$.*

In order to prove Theorem 3.3, we need the following important theorem on gluing plurisubharmonic functions.

Theorem 3.4. *Let Ω be a strictly pseudoconvex domain in \mathbb{C}^n with \mathcal{C}^2 boundary. Let $u \in \mathcal{C}^1(\overline{\Omega}) \cap PSH(\Omega)$ and $v \in \mathcal{C}^1(\mathbb{C}^n \setminus \Omega) \cap PSH(\mathbb{C}^n \setminus \overline{\Omega})$, such that $u = v$ on the boundary of Ω . If the normal derivatives satisfy $\frac{\partial u}{\partial n} \leq \frac{\partial v}{\partial n}$ on the boundary of Ω , then the function defined by*

$$V := \begin{cases} u & \text{on } \Omega; \\ v & \text{on } \mathbb{C}^n \setminus \Omega \end{cases}$$

is plurisubharmonic on \mathbb{C}^n .

Here n is the outward unit normal and the normal derivative is $\frac{\partial u}{\partial n} = \nabla u \cdot n$. Blanchet and Gauthier give the proof of the above theorem for subharmonic functions and domains with \mathcal{C}^1 boundary in [4]. Moreover, they state this theorem for subsolutions of any elliptic partial differential equation. This theorem is true in a more general setting; here we give

an elementary proof for the case of plurisubharmonic functions in strictly pseudoconvex domains and we do not use the theory of distributions.

Proof. Upper semicontinuity of V is trivial as it is continuous. Since u and v are plurisubharmonic on Ω and $\mathbb{C}^n \setminus \overline{\Omega}$ respectively, we have V is plurisubharmonic on $\mathbb{C}^n \setminus \partial\Omega$. Thus, we need to show that at every point $z \in \partial\Omega$, we have V is subharmonic on each complex line passing through z .

We fix $z \in \partial\Omega$. Since plurisubharmonicity is preserved under biholomorphic mappings, applying a translation and a rotation, we may assume that $z = 0$ and the outer normal, n_1 , is the real vector $n_1 = (1, 0, \dots, 0) \in \mathbb{R}^{2n}$. Furthermore, we may take $u(0) = v(0) = 0$.

By the Narasimhan lemma (See [9] Lemma 3.2.3), there is a neighborhood U of 0 and a biholomorphic mapping of U , $\varphi : U \rightarrow \mathbb{C}^n$, such that $\varphi(U \cap \Omega)$ is strictly convex. By applying such a biholomorphic mapping of a neighborhood of the origin, we may assume that Ω is strictly convex around this point. By the proof of the Narasimhan lemma, the Jacobian matrix of φ at $z = 0$ can be taken as the identity matrix. Therefore, the unit normals, the normal derivatives and the inequality $\frac{\partial u}{\partial n}(0) \leq \frac{\partial v}{\partial n}(0)$ are preserved under the mapping.

Note that u and v can be extended to a neighborhood of the origin such that the extensions are differentiable at the origin. We will continue writing u and v for the extensions and note that our calculations are independent of the extensions. Now we will show that V satisfies the following integral inequality on any complex line passing through 0.

$$V(0) \leq \frac{1}{2\pi} \int_0^{2\pi} V(0 + b e^{i\theta}) d\theta$$

where $b = \rho((a_1, b_1), \dots, (a_n, b_n))$ so that

$$b e^{i\theta} = \rho((a_1 \cos(\theta), b_1 \sin(\theta)), \dots, (a_n \cos(\theta), b_n \sin(\theta))).$$

We may also assume that a_1 is positive. Otherwise, we may take $-b$ which gives the same complex line.

If the complex line intersects with Ω only at 0, i.e., the complex line lies entirely in the tangent space, then consider $z_j := (x_j, 0, \dots, 0) \in \mathbb{R}^{2n}$ where $x_j > 0$ such that $z_j \rightarrow 0$. Note that all $z_j \in \mathbb{C}^n \setminus \overline{\Omega}$ and the complex line passing through z_j which is in the direction of b lies completely in $\mathbb{C}^n \setminus \overline{\Omega}$.

By plurisubharmonicity of v on $\mathbb{C}^n \setminus \overline{\Omega}$, we have

$$V(z_j) = v(z_j) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_j + b e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} V(z_j + b e^{i\theta}) d\theta.$$

As $z_j \rightarrow 0$, we have $v(z_j) \rightarrow v(0)$. By uniform continuity of v in a neighborhood of 0, we have

$$\int_0^{2\pi} V(z_j + b e^{i\theta}) d\theta \rightarrow \int_0^{2\pi} V(0 + b e^{i\theta}) d\theta.$$

Hence, we obtain

$$V(0) \leq \frac{1}{2\pi} \int_0^{2\pi} V(0 + b e^{i\theta}) d\theta.$$

If the complex line intersects with Ω not only at 0, then using the fact that Ω is strictly convex in a neighborhood of 0 we can write

$$\int_0^{2\pi} V(b e^{i\theta}) d\theta = \int_{\theta_1}^{\theta_2} u(b e^{i\theta}) d\theta + \int_{\theta_2}^{\theta_1} v(b e^{i\theta}) d\theta,$$

where $\pi/2 < \theta_1 < \pi$ and $\pi < \theta_2 < 3\pi/2$. Now using the total differential of u and v around 0 we have

$$(3.1) \quad u(z) = \sum_{i=1}^n u_{x_i}(0)x_i + \sum_{i=1}^n u_{y_i}(0)y_i + \sum_{i=1}^n \varepsilon_i x_i + \sum_{i=1}^n \varepsilon'_i y_i,$$

$$(3.2) \quad v(z) = \sum_{i=1}^n v_{x_i}(0)x_i + \sum_{i=1}^n v_{y_i}(0)y_i + \sum_{i=1}^n \eta_i x_i + \sum_{i=1}^n \eta'_i y_i,$$

where $\varepsilon_i, \eta_i \rightarrow 0$ as $x_i \rightarrow 0$ and $\varepsilon'_i, \eta'_i \rightarrow 0$ as $y_i \rightarrow 0$. We remark that all first order partial derivatives of u and v vanish except the partial derivatives with respect to x_1 . This is due to the facts that $(1, 0, \dots, 0)$ is the outer normal and the rest of the standard basis elements lie in the tangent space and that u and v are defining functions for the boundary of Ω . By defining $A := u_{x_1}(0) = \frac{\partial u}{\partial n}(0)$ and $B := v_{x_1}(0) = \frac{\partial v}{\partial n}(0)$, we note that $A \leq B$. Thus, we have

$$\int_0^{2\pi} V(b e^{i\theta}) d\theta = \int_{\theta_1}^{\theta_2} u(b e^{i\theta}) d\theta + \int_{\theta_2}^{\theta_1} v(b e^{i\theta}) d\theta,$$

which equals to

$$\begin{aligned} & \int_{\theta_1}^{\theta_2} \rho \left(u_{x_1}(0)(a_1 \cos(\theta)) + \sum_{i=1}^n \varepsilon_i(a_i \cos(\theta)) + \sum_{i=1}^n \varepsilon'_i(b_i \sin(\theta)) \right) d\theta \\ & + \int_{\theta_2}^{\theta_1} \rho \left(v_{x_1}(0)\rho(a_1 \cos(\theta)) + \sum_{i=1}^n \eta_i(a_i \cos(\theta)) + \sum_{i=1}^n \eta'_i(b_i \sin(\theta)) \right) d\theta, \end{aligned}$$

which is greater than or equal to

$$M := (A - B)\rho[\sin(\theta_2) - \sin(\theta_1)] - 2\pi\rho[\sup(\varepsilon_i, \eta_i, \varepsilon'_i, \eta'_i)]\left[\max_{i=1,\dots,n} \{|a_i|, |b_i|\}\right].$$

Note that M is positive by the facts that $A - B \leq 0$, that $\sin(\theta_2) - \sin(\theta_1) < 0$, that $\varepsilon_i, \eta_i \rightarrow 0$ as $x_i \rightarrow 0$ and $\varepsilon'_i, \eta'_i \rightarrow 0$ as $y_i \rightarrow 0$. So $M \geq 0 = V(0)$. Hence, V satisfies the integral inequality on each complex line. Therefore, V is plurisubharmonic. \square

Proof of Theorem 3.3. Since Ω is a strictly pseudoconvex domain, there exists a twice continuously differentiable strictly plurisubharmonic defining function ρ defined in a neighborhood of the closure of Ω . By Theorem 1.1 of [6], the extremal function V_K is in $\mathcal{C}^{1,1}(\mathbb{C}^n \setminus \Omega)$. Now we define $\rho_\varepsilon := \varepsilon\rho$. Then, there exists $\varepsilon_0 > 0$ that $\frac{\partial \rho_{\varepsilon_0}}{\partial n} < \frac{\partial V_K}{\partial n}$ on $\partial\Omega$.

We define $Q := \rho_{\varepsilon_0}$. Now consider

$$(3.3) \quad V := \begin{cases} Q & \text{on } \Omega, \\ V_K & \text{on } \mathbb{C}^n \setminus \Omega. \end{cases}$$

By Theorem 3.4, V is plurisubharmonic and it is in Lelong class L^+ . Since $V = Q$ on K and V_K is maximal on $\mathbb{C}^n \setminus \Omega$, by domination principle, we have $V_{K,Q} = V$ and the support $\text{supp}(dd^c V_{K,Q})^n = \overline{\Omega}$ by definition of Q . \square

The following observation gives us a way of obtaining global subharmonic functions in Lelong class L^+ from a harmonic function defined in the unit disc (denoted by Δ) of \mathbb{C} .

Remark 3.5. Let h be a harmonic function in the unit disc of \mathbb{C} which is in $C^1(\overline{\Delta})$. If $|\frac{\partial h}{\partial n}| \leq 1/2$, then the function defined by

$$(3.4) \quad g(z) = \begin{cases} h(z), & \text{if } z \in \overline{\Delta}; \\ h(\frac{1}{\bar{z}}) + \log|z|, & \text{otherwise,} \end{cases}$$

is subharmonic and in L^+ , where n is the outer normal. From this we can easily obtain a family of examples in the disc which is a special case of a result of Varju and Totik [15].

This observation gives us a way of constructing nonconstant weights on $p\Delta$ such that the support of the weighted extremal measure is $p\Delta$.

Let K be the boundary of a bounded domain Ω with smooth boundary. Our goal is to find a continuous weight, Q , such that the support of $(dd^c V_{K,Q})^n$ is the whole boundary. Note that if Ω is strictly pseudoconvex, then constant weights are sufficient for this purpose. However, we want to find examples of nonconstant weights satisfying this condition. One may

hope that if $V_{K,Q} = Q$ on $\partial\Omega$, then the support of the Monge-Ampère measure is $\partial\Omega$.

The following example shows that for a given nonconstant weight Q on $\partial\Omega = K$, the support of $(dd^c V_{K,Q})^n$ is not necessarily all of $\partial\Omega$. However, $V_{K,Q} = Q$ on K .

Example 3.6. Let K be the boundary of a bounded domain with smooth boundary. Let L be a proper subset of K which is regular and the polynomially convex hull of L is a proper subset of K . By regularity of L , we have V_L is continuous. Moreover, it is not constant on K as it is not the polynomially convex hull of K .

We define $Q : K \rightarrow \mathbb{R}$ by $Q(z) = V_L(z)$. Now $V_{K,Q} = V_L$ whose Monge-Ampère measure is supported on L , which is a proper subset of K .

The next example shows that any closed ball can be obtained as the support of a Monge-Ampère measure. Although this example is well known, we include it here for the sake of completeness.

Example 3.7. Let $K := \overline{B(0, R)}$. We define $Q(z) = A(|z|^2 - R^2)$ where $2AR \leq 1$. Then by Theorem 3.4, we get

$$V_{K,Q}(z) = \begin{cases} A(|z|^2 - R^2), & \text{if } |z| \leq R; \\ \log |z| - \log R, & \text{if } |z| > R. \end{cases}$$

It is clear that the support of the Monge-Ampère measure is K .

The following example shows that the finite union of concentric spheres can be obtained as the support of a weighted equilibrium measure.

Example 3.8. Let $K_m = \overline{B(0, r_m)}$ for some $r_m > 0$. We will inductively define weights on K_m such that the weighted equilibrium measure is supported on the concentric spheres, $\bigcup_{i=1}^m \partial B(0, r_i)$ for $m \geq 1$ and $0 < r_1 < r_2 < \dots < r_m$.

Let $m = 1$, we define $Q_1(z) = \log r_1$. Clearly, $V_{K_1, Q_1}(z) = \max(\log |z|, \log r_1)$ and $(dd^c V_{K_1, Q_1})^n$ is supported on the sphere of radius r_1 with center origin.

For $m \geq 2$, we define $Q_m = \frac{1}{2}V_{K_{m-1}, Q_{m-1}}|_{K_m}$. We remark that each Q_m is continuous and K_m is locally regular, hence, each of V_{K_m, Q_m} is continuous by Proposition 2.13 of [14]. Since each weight function and the weighted extremal function are radial, we can write $V_{K_m, Q_m}(z) = V_{K_m, Q_m}(|z|)$. We will show that

$$V_{K_m, Q_m}(z) = \begin{cases} \frac{1}{2}V_{K_{m-1}, Q_{m-1}}(z), & \text{if } |z| \leq r_m; \\ \log |z| + A_m, & \text{if } |z| > r_m, \end{cases}$$

where $A_m = \frac{1}{2}V_{K_{m-1}, Q_{m-1}}(r_m) - \log r_m$.

If the function v_m , on the right hand side, is plurisubharmonic then it is maximal outside of K_m . In addition, using the fact that v_m equals to Q_m on K_m , we have $V_{K_m, Q_m} = v_m$ by the domination principle. Thus, it is enough to show that the function v_m is plurisubharmonic.

For $m = 2$, we have $\frac{1}{2}V_{K_1, Q_1} = \frac{1}{2} \max(\log |z|, \log r_1)$. Since

$$\frac{\partial \left\{ \frac{1}{2} \max(\log |z|, \log r_1) \right\}}{\partial n} < \frac{\partial \{ \log |z| + A_2 \}}{\partial n}$$

on $\partial B(0, r_2)$, by Theorem 3.4, we have that v_2 is plurisubharmonic.

For $m > 2$, by induction hypothesis we have $V_{K_m, Q_m}(z) = \log |z| + A_{m-1}$ on $B(0, r_m) \setminus B(0, r_{m-1})$. Hence, again by Theorem 3.4, we obtain that v_m is plurisubharmonic.

Next, we show that $\text{supp}(dd^c V_{K_m, Q_m})^n = \bigcup_{i=1}^m \partial B(0, r_i)$.

For $m = 1$, this is obvious. For $m \geq 2$,

$$V_{K_m, Q_m}(z) = \begin{cases} \frac{1}{2}V_{K_{m-1}, Q_{m-1}}, & \text{if } |z| \leq r_m; \\ \log |z| + A_m, & \text{if } |z| > r_m. \end{cases}$$

Clearly, V_{K_m, Q_m} is maximal on $\mathbb{C}^n \setminus K_m$. Hence, $\text{supp}(dd^c V_{K_m, Q_m})^n \subset K_m$. Since $V_{K_m, Q_m}(z) = \frac{1}{2}(\log |z| + A_{m-1})$ on $B(0, r_m) \setminus B(0, r_{m-1})$, we have

$$\partial B(0, r_m) \subset \text{supp}(dd^c V_{K_m, Q_m})^n.$$

On $B(0, r_m)$, we have $V_{K_m, Q_m} = \frac{1}{2}V_{K_{m-1}, Q_{m-1}}$. By induction assumption, we have

$$\text{supp}(dd^c V_{K_{m-1}, Q_{m-1}})^n = \bigcup_{i=1}^{m-1} \partial B(0, r_i).$$

Therefore, we have $\text{supp}(dd^c V_{K_m, Q_m})^n = \bigcup_{i=1}^m \partial B(0, r_i)$.

Note that in this example, we can take $K_m = \bigcup_{i=1}^m \partial B(0, r_i)$ and similar arguments will give the same conclusion.

The next example shows that shells, i.e., the difference of two concentric balls, can be obtained as the support of the Monge-Ampère measure of a weighted extremal function.

Example 3.9. Let $K = \overline{B(0, R)} \setminus B(0, r)$ where $r < R$. Let

$$Q(z) = \frac{1}{R-r} (|z| - r \log |z| - r + r \log R).$$

Later we will verify that

$$V_{K, Q}(z) = \begin{cases} 0, & \text{if } |z| < r; \\ \frac{1}{R-r} (|z| - r \log |z| - r + r \log R), & \text{if } r \leq |z| \leq R; \\ \log |z| - \log R + \frac{1}{R-r} (R - r \log R - r + r \log R), & \text{if } |z| > R \end{cases}$$

and that the support of the Monge-Ampère measure of the weighted extremal function is the shell $K = \overline{B(0, R)} \setminus B(0, r)$.

Our next goal is to show that many radially symmetric compact sets can be obtained as the support of the Monge-Ampère measure of a weighted extremal function $V_{K,Q}^*$, for some admissible continuous weight Q .

To obtain results on radially symmetric compact sets and in order to verify Example 3.9, we recall some facts about radially symmetric plurisubharmonic functions and the representation of these functions in terms of their Monge-Ampère measures from Persson's thesis [11]. We note that this work is not published anywhere else.

Proposition 3.10. [11, Proposition 3.1] *Let $u(z) = \tilde{u}(\log |z|^2)$ be an upper semicontinuous function. Then u is plurisubharmonic on $B(0, e^R)$ if and only if \tilde{u} is increasing on $[-\infty, R)$ and convex on $(-\infty, R)$, for $-\infty < R \leq \infty$.*

Persson computes the entries of the complex Hessian matrix of a radially symmetric plurisubharmonic function $u(z) := \tilde{u}(\log |z|^2) \in \mathcal{C}^2$ and obtains

$$(3.5) \quad \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = \delta_{ij} \frac{\tilde{u}'(\log |z|^2)}{|z|^2} + \bar{z}_i z_j \frac{\tilde{u}''(\log |z|^2) - \tilde{u}'(\log |z|^2)}{|z|^4}.$$

Using linear algebra, he obtains the Monge-Ampère measure of u as

$$(dd^c u)^n = \frac{(\tilde{u}'(\log |z|^2))^{n-1} \tilde{u}''(\log |z|^2)}{|z|^{2n}} dV(z),$$

where dV is the standard volume form on \mathbb{C}^n . By direct integration, he obtains the following representation theorem.

Theorem 3.11. [11, Corollary 3.4] *Let u be a radial plurisubharmonic function bounded from below, then we have the following representation formula.*

$$(3.6) \quad u(z) = u(0) + \int_{0 < |\zeta| < |z|} \frac{|\zeta|^{-2n}}{n\omega_{2n}} \left(\frac{(dd^c u)^n(B(0, |\zeta|))}{4^n n! \omega_{2n}} \right)^{1/n} dV(\zeta).$$

Conversely, if μ is a radially symmetric nonnegative measure with compact support satisfying

$$(3.7) \quad \int_{B(0,r)} |\zeta|^{-2n} (\mu(B(0, |\zeta|)))^{1/n} dV(\zeta) < \infty,$$

then

$$(3.8) \quad u(z) = \int_{0 < |\zeta| < |z|} \frac{|\zeta|^{-2n}}{n\omega_{2n}} \left(\frac{\mu(B(0, |\zeta|))}{4^n n! \omega_{2n}} \right)^{1/n} dV(\zeta)$$

defines a radially symmetric plurisubharmonic function bounded from below and $(dd^c u)^n = \mu$.

As a by-product of the proof of Theorem 3.3 of [11], we get the following one dimensional integral representation for radially symmetric plurisubharmonic functions which are bounded from below [11, page 15].

$$(3.9) \quad u(z) = u(0) + \int_0^{|z|} \frac{2}{t} \left(\frac{\mu(B(0,t))}{4^n n! \omega_{2n}} \right)^{1/n} dt.$$

Given a radially symmetric compact set K , we are going to construct an appropriate positive radially symmetric measure whose support is K and which satisfies (3.7). Then we will use the representation (3.9) to construct a radially plurisubharmonic function in the Lelong class L^+ with Monge-Ampère measure supported on K .

Given a radially symmetric μ satisfying (3.7), we make the following observations. Here, we use the notation $f(t) := \mu(B(0,t))$.

- (1) f is a nondecreasing function of t .
- (2) $f(t)$ is constant for $t > T_0$ for some $T_0 > 0$ if and only if μ has compact support.
- (3) Let μ be a measure with compact support, i.e., $f(t)$ is constant for $t > T_0$, then $u \in L^+$ if and only if $2 \left(\frac{\mu(B(0,T_0))}{4^n n! \omega_{2n}} \right)^{1/n} = 1$, i.e., $\mu(B(0,T_0)) = (2\pi)^n$. This condition comes from equation (3.9) and it is a necessary condition for u to be in L^+ .

As an immediate application of the above formula, we verify Example 3.9, where we obtain shells as support of the weighted equilibrium measure.

We define the following radial measure

$$f(t) := \mu(B(0,t)) = \begin{cases} 0, & \text{if } t < r; \\ \left(\frac{2\pi}{R-r} (t-r) \right)^n, & \text{if } r < t < R; \\ (2\pi)^n, & \text{if } t > R. \end{cases}$$

Assuming $u(0) = 0$, we obtain the corresponding function defined by (3.9) as:

$$u(z) = \begin{cases} 0, & \text{if } |z| < r; \\ \frac{1}{R-r} (|z| - r \log |z| - r + r \log R), & \text{if } r < |z| < R; \\ \log |z| - \log R + \frac{1}{R-r} (R - r \log R - r + r \log R), & \text{if } |z| > R, \end{cases}$$

which is a continuous plurisubharmonic function. Note that u is maximal off K and $u \in L^+$. Since $Q = u|_K$, we obtain $V_{K,Q} = u$.

In fact, we get a similar construction by taking any continuous nondecreasing function $f(t)$ of the form

$$(3.10) \quad f(t) := \mu(B(0,t)) = \begin{cases} 0, & \text{if } t < r; \\ g(t), & \text{if } r < t < R; \\ (2\pi)^n, & \text{if } t > R, \end{cases}$$

where $g(t)$ is a strictly increasing function such that $g(r) = 0$ and $g(R) = (2\pi)^n$.

We remark that we can obtain the case of $m = 1$ in Example 3.8 by using a measure μ of the form

$$(3.11) \quad \mu(B(0, t)) = \begin{cases} 0, & \text{if } t < r_1; \\ (2\pi)^n, & \text{if } t \geq r_1. \end{cases}$$

Thus, by (3.9), we obtain $V_{K,Q}(z) = \max\{0, \log |z| - \log r_1\}$.

A similar construction works with a compact set K which is a countable union of spheres and closed shells. Let $K := \bigcup_{i=1}^{\infty} K_i$ where each K_i is either

a sphere or an annulus. We define a measure $\mu := \sum_{i=1}^{\infty} \frac{\mu_i}{2^i}$ where each μ_i is supported on each K_i . If K_i is a sphere, then μ_i is of the form (3.11) and if K_i is an annulus, then μ_i is of the form (3.10). By using (3.9) we obtain a plurisubharmonic function $u \in L^+$ such that the support of the Monge-Ampère measure of u is K . Hence, by defining $Q = u|_K$, we obtain the desired result.

4. OPEN PROBLEMS

In this section, we list some open problems related to supports of extremal measures in connection with supports of Monge-Ampère measures of plurisubharmonic functions in Lelong class.

Open Problem 4.1. *Under which conditions weighted extremal measures $(dd^c V_{K,Q}^*)^n$ and $(dd^c U_{K,Q,\Omega}^*)^n$ are mutually absolutely continuous? In particular, if their supports are equal, are they mutually absolutely continuous?*

Note that Example 2.8 and Example 2.9 show that continuous weights and plurisubharmonic weights are not enough to have mutual absolute continuity of the extremal functions. The conclusion holds for constant weights obviously.

Open Problem 4.2. *Find conditions on K and Q such that $V_{K,Q} = Q$ on K . Note that in this case Q should be continuous plurisubharmonic in the interior of K .*

Open Problem 4.3. *Under which conditions a compact set $K \subset \mathbb{C}^n$ can be written as the support of a weighted extremal function? Namely, which sets are the support of the Monge-Ampère measure of a continuous plurisubharmonic function in Lelong class, L^+ ?*

Note that, in this case the set K cannot be pluripolar at any point.

Open Problem 4.4. Which (compactly supported) measures are the Monge-Ampère measures of continuous plurisubharmonic functions in Lelong class?

The relation between this problem and the previous one is: If there exists $u \in L^+ \cap \mathcal{C}(\mathbb{C}^n)$ such that $\text{supp}(dd^c u)^n = K$ then we can define $Q = u|_K$ which is an admissible weight and by the domination principle $V_{K,Q} = u$.

This question is equivalent to finding the range of $T : L^+ \cap \mathcal{C}(\mathbb{C}^n) \rightarrow \mathcal{PL}(\mathbb{C}^n)$, where $T(u) = (dd^c u)^n$ and $\mathcal{PL}(\mathbb{C}^n)$ is the set of Borel measures with compact support and total mass $(2\pi)^n$. In this case if μ is in the range of T , then μ has no mass on pluripolar sets.

Note that Guedj and Zeriahi showed that if μ is a measure which puts no mass on pluripolar sets and has total mass $(2\pi)^n$, then there exists $u \in L^+$ such that $(dd^c u)^n = \mu$. See [7].

Also note that for $n = 1$, Arsove answered this question in terms of density of the measure. However, again the notion of logarithmic potentials is used. See [1] for details.

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